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A REAL VARIABLE RESTATEMENT OF RIEMANN'S HYPOTHESIS

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ABSTRACT
We show that Riemann's hypothesis is related to the equality of certain interesting subspaces of $L^p(0,1)$. Our results generalize an earlier theorem of A. Beurling [2].

Let C denote the linear space generated by the functions $\{\rho_\theta : 0 < \theta \leqq 1\}$ defined by

$$\rho_\theta(x) = \rho(\theta/x) - \theta\rho(1/x), \qquad 0 < x \leqq 1,$$

where $\rho(x) = x - [x]$ represents the fractional part of the real number x. If C^p denotes the closure of C in $L^p(0,1)$, we have the following result due to A. Beurling [2].

THEOREM A. *The Riemann Zeta-function has no zeros s with* $\operatorname{Re} s > 1/p$ *if and only if* $C^p = L^p(0,1)$.

Beurling's original proof used a theorem on the existence in $L^q(0,1)$ of characters of the multiplicative semigroup $(0,1)$, orthogonal to certain subspaces of $L^p(0,1)$ $(1/q + 1/p = 1)$.

In the present paper we give an alternative approach to Beurling's theorem, using the harmonic analysis of the semigroup $\{V_2(t) : t \geqq 0\}$ of unitary operators on $L^2(0,1)$, defined by

$$(V_2(t)f)(x) = e^{t/2} f(e^t x), \qquad x \in [0, e^{-t}],$$
$$= 0, \qquad x \in (e^{-t}, 1].$$

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This approach provides a better understanding of the infinite dimensional geometry involved in the study of Riemann's hypothesis. Our analysis also yields alternative formulations, worth studying, of the problem.

For example, let Z denote the linear space of all functions f that can be represented as

$$f(x)=\sum_{k=1}^{\infty}\{F(2kx)-F((2k-1)x)\}$$

for some continuous function F of bounded variation on $[0,\infty)$, such that $F(x)=0$ for $x \geqq 1$. If Z^p denotes the closure of Z in $L^p(0,1)$, the following theorem is a particular case of our results (cf. Section 3 below).

THEOREM B. *For* $1 \leqq p \leqq 2$ *the Riemann Zeta-function has no zeros* s *with* $\operatorname{Re} s > 1/p$ *if and only if* $Z^p = Z^1 \cap L^p(0,1)$.

The fact that the spaces Z^p might play a more basic role, in connection with Riemann's hypothesis, than Beurling's spaces C^p was suggested to us by Professor Max Zorn to whom (as well as to Professor James P. Williams) we are indebted for our interest in Beurling's paper [2]. Therefore, we suggest to call Z^p the Zorn subspace of $L^p(0,1)$ and we hope that Theorem B shows that these Zorn subspaces are interesting objects of real analysis.

1. Preliminaries

Let $\{T(t): t > 0\}$ be a strongly continuous semigroup of isometries acting on a Hilbert space H. It is then known (cf. [7], Ch. III) that there exists an isometry T, called the cogenerator of $\{T(t): t \geqq 0\}$, such that

$$T(t)=\exp\left[t(T+I)(T-I)^{-1}\right], \qquad t \geqq 0.$$

Moreover, if A is the generator of $\{T(t): t \geqq 0\}$, i.e.,

$$Ah=\lim_{t\to 0+}\left[T(t)h-h\right]/t$$

for all h in H for which the limit exists, then

$$T=I+2(A-I)^{-1}.$$

It easily follows from these formulas that a closed subspace M of H is invariant under T if and only if it is invariant under $T(t)$ for all $t \geqq 0$.

For every measurable function f on $[0,1]$ we set

$$(V(t)f)(x) = e^{t/2} f(e^t x), \qquad x \in [0, e^{-t}],$$
$$= 0, \qquad x \in (e^{-t}, 1).$$

LEMMA 1.1. *For $f \in L^p(0,1)$ we have*

$$\| V(t)f \|_p = \exp\left[t\left(\frac{1}{2} - \frac{1}{p} \right) \right] \| f \|_p.$$

Moreover, the restrictions $V_2(t)$ of $V(t)$ to $L^2(0,1)$ form a strongly continuous semigroup of isometries on $L^2(0,1)$.

PROOF. The estimate of $\| V(t)f \|_p$ is a straightforward computation. That $\{V_2(t) : t \geqq 0\}$ is strongly continuous follows from the fact that $V_2(t)$ is an isometry combined with the fact that $V_2(t)f$ converges uniformly to f as $t \to 0+$ if f is continuous with compact support in $[0, 1)$.

Let H^2 denote, as usual, the Hilbert space of functions $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ defined for $|\lambda| < 1$, and with finite norm

$$\| f \|_{H^2} = \left(\sum_{n=0}^{\infty} | a_n |^2 \right)^{1/2}.$$

On H^2 consider the semigroup of isometries $\{S(t) : t \geqq 0\}$ with cogenerator S given by the formulas

$$(S(t)f)(\lambda) = \exp\left(t \frac{\lambda + 1}{\lambda - 1} \right) f(\lambda),$$

$$(Sf)(\lambda) = \lambda f(\lambda), \qquad |\lambda| < 1.$$

LEMMA 1.2. *There exists a unitary operator $\mathcal{F} : L^2(0,1) \to H^2$ such that $S(t)\mathcal{F} = \mathcal{F} V_2(t)$, $t \geqq 0$, and $S\mathcal{F} = \mathcal{F} V_2$.*

PROOF. We obviously have $\bigcap_{t \geqq 0} V_2(t) L^2(0,1) = \{0\}$ and this implies that $V_2(t)$ and V_2 are completely nonunitary isometries, i.e., forward shifts (cf. [7], Ch. III). Moreover, $\bigvee_{t \geqq 0} V_2(t)\mathbf{1} = L^2(0,1)$, where $\mathbf{1}$ denotes the constant function identically equal to 1. Indeed, the function $e^{-t/2} V_2(t)\mathbf{1}$ coincides with the characteristic function of $[0, e^{-t}]$ and these characteristic functions generate $L^2(0,1)$. Thus we also have $\bigvee_{n \geqq 0} V_2^n \mathbf{1} = L^2(0,1)$ so that V_2 has a cyclic vector and hence it is a forward shift of multiplicity one. Since every forward shift of multiplicity one is unitarily equivalent to S, we infer the existence of a unitary operator $\mathcal{F} : L^2(0,1) \to H^2$ such that $S\mathcal{F} = \mathcal{F} V_2$. The relations $S(t)\mathcal{F} = \mathcal{F} V_2(t)$ follow now from the formula relating a semigroup with its cogenerator.

In order to find an explicit formula for $\mathcal{F}$, we introduce the functions p_μ in H^2

defined for $|\mu|<1$ by $p_\mu(\lambda)=(1-\bar{\mu}\lambda)^{-1}$, $|\lambda|<1$. We have $(f,p_\mu)=f(\mu)$ and $S^*p_\mu=\bar{\mu}p_\mu$ for f in H^2 and $|\mu|<1$ so that

$$S(t)^*p_\mu=\exp\left(t\frac{\bar{\mu}+1}{\bar{\mu}-1}\right)p_\mu.$$

Also note that p_μ is antianalytic in μ and $\|p_\mu\|_{H^2}=(1-|\mu|^2)^{-1/2}$.

LEMMA 1.3. *If $\mathcal{F}$ is the unitary operator from Lemma 1.2, there exists a complex constant α, $|\alpha|=1$, such that*

$$(\mathcal{F}f)(\mu)=\frac{\alpha\sqrt{2}}{1-\mu}\int_0^1 f(x)x^{-1/2+(1+\mu)/(1-\mu)}dx, \tag{1.1}$$
$$f\in L^2(0,1),\quad |\mu|<1.$$

PROOF. We have

$$(\mathcal{F}f)(\mu)=(\mathcal{F}f,p_\mu)=(f,\mathcal{F}^*p_\mu)$$

and therefore it will suffice to show that

$$(\mathcal{F}^*p_\mu)(x)=\frac{\alpha\sqrt{2}}{1-\mu}x^{-1/2+(1+\bar{\mu})/(1-\bar{\mu})}$$

for some α independent of μ, $|\alpha|=1$. The relations $V_2(t)^*\mathcal{F}^*=\mathcal{F}^*S(t)^*$ imply that

$$V_2(t)^*(\mathcal{F}^*p_\mu)=\exp\left(t\frac{\bar{\mu}+1}{\bar{\mu}-1}\right)(\mathcal{F}^*p_\mu),\qquad t\geqq 0. \tag{1.2}$$

The adjoint $V_2(t)^*$ is given by

$$(V_2(t)^*f)(x)=e^{-t/2}f(e^{-t}x),\qquad x\in[0,1]$$

so that (1.2) reduces to

$$e^{-t/2}(\mathcal{F}^*p_\mu)(e^{-t}x)=\exp\left(t\frac{\bar{\mu}+1}{\bar{\mu}-1}\right)(\mathcal{F}^*p_\mu)(x) \tag{1.3}$$

and this easily implies

$$(\mathcal{F}^*p_\mu)(x)=c_\mu x^{-1/2+(1+\bar{\mu})/(1-\bar{\mu})}$$

where c_μ does not depend on x (formally we set $x=1$ and $e^{-t}=y$ in (1.3) to get $c_\mu=(\mathcal{F}^*p_\mu)(1)$; this formal computation is easy to justify).

A direct computation shows that

$$\| x^{-1/2+(1+\bar{\mu})/(1-\bar{\mu})} \|_2 = \frac{|1-\bar{\mu}|}{\sqrt{2}}(1-|\mu|^2)^{-1/2}$$

so that the equality $\| \mathcal{F}^* p_\mu \| = \| p_\mu \| = (1-|\mu|^2)^{-1/2}$ gives

$$c_\mu = \frac{\alpha_\mu \sqrt{2}}{1-\bar{\mu}}, \qquad |\alpha_\mu| = 1.$$

We use now the fact that $\mathcal{F}^* p_\mu$ is antianalytic in μ so that α_μ must be antianalytic in μ. This clearly implies that α_μ does not depend on μ. The lemma is proved.

Note that $\mu \to (1+\mu)/(1-\mu)$ maps the unit disc D onto the halfplane $\{\lambda : \operatorname{Re} \lambda > 0\}$ so that the formula $s = (1+\mu)/(1-\mu) + \frac{1}{2}$ establishes a conformal correspondence between the unit disc $D = \{\mu : |\mu| < 1\}$ and the halfplane $\{s : \operatorname{Re} s > \frac{1}{2}\}$. Since $s + \frac{1}{2} = 2/(1-\mu)$ formula (1.1) can be rewritten as

$$(1.4) \qquad (\mathcal{F}f)(\mu) = \frac{\alpha}{\sqrt{2}}(s+\tfrac{1}{2}) \int_0^1 f(x) x^{s-1} dx, \qquad s = \frac{1+\mu}{1-\mu} + \tfrac{1}{2}.$$

For the functions $\rho_\theta(x) = \rho(\theta/x) - \theta\rho(1/x)$, defined in the introduction, a direct calculation shows that

$$(1.5) \qquad (\mathcal{F}\rho_\theta)(\mu) = \frac{\alpha}{\sqrt{2}}(s+\tfrac{1}{2}) \frac{\theta - \theta^s}{s} \zeta(s), \qquad s = \frac{1+\mu}{1-\mu} + \tfrac{1}{2}$$

for $\operatorname{Re} s > 1$ (cf. [2] and [3]). Formula (1.5) extends by analytic continuation to the entire domain $\{s : \operatorname{Re} s > \frac{1}{2}\}$.

Observe that (1.5) is (equivalent to) the basic formula in Beurling's paper [2].

Remark 1.4. The following explicit formula for the cogenerator V_2 of the semigroup $\{V_2(t) : t \geqq 0\}$ can be easily obtained by using (1.1) and the relation $S\mathcal{F} = \mathcal{F}V_2$:

$$(1.6) \qquad (V_2 f)(x) = f(x) - \int_x^1 2x^{1/2} y^{-3/2} f(y) dy, \qquad f \in L^2(0,1).$$

Formula (1.6) can also be obtained directly, from the definition of cogenerators.

2. The main result

We denote by Z_θ the linear space generated by the functions $\{V(t)\rho_\theta : t \geqq 0\}$. Also, following Beurling, we denote by C the linear space generated by $\{\rho_\theta : 0 < \theta < 1\}$. The formula

$$V(t)\rho_\theta = e^{t/2}\rho_{\theta e^{-t}} - \theta e^{t/2}\rho_{e^{-t}}$$

shows that C is invariant under $V(t)$ so that $Z_\theta \subset C$ for $0 < \theta < 1$. Denote by Z_θ^p [resp. C^p] the closure of Z_θ [resp. C] in $L^p(0,1)$ and note that the spaces Z_θ^p and C^p are invariant under $V(t)$ for all $t \geqq 0$.

In what follows, we will use the notation

$$A_\theta = \{s : \operatorname{Re} s > \tfrac{1}{2} \text{ and } \zeta(s) = 0\} \cup \left\{ s = 1 + \frac{2k\pi i}{\log\theta} : k \in \mathbf{Z} \setminus \{0\} \right\}.$$

For s in A_θ with $\zeta(s) = 0$, we denote by $n(s)$ the multiplicity of s as zero of ζ. For $s = 1 + 2k\pi i/\log\theta$ we set $n(s) = 1$; this makes sense since by a celebrated theorem of Hadamard and de la Vallée-Poussin (cf., e.g., [8]) $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$.

If g is an analytic function of s, the notation

$$g(s) = O((s - s_0)^n)$$

will indicate that g has a zero of order at least n at s_0.

PROPOSITION 2.1. *For $0 < \theta < 1$ we have*

$$Z_\theta^2 = \left\{ f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s - s_0)^{n(s_0)}) \text{ for all } s_0 \text{ in } A_\theta \right\}.$$

PROOF. The subspace Z_θ^2 of $L^2(0,1)$ is invariant under $V_2(t)$, $t \geqq 0$, and therefore under V_2. The relation $S\mathcal{F} = \mathcal{F}V_2$ shows that $\mathcal{F}Z_\theta^2$ is invariant under S. By a theorem of Beurling [1] there exists an inner function m_θ in H^∞ such that $\mathcal{F}Z_\theta^2 = m_\theta H^2$. Now, Z_θ^2 is the smallest invariant subspace for V_2 containing the function ρ_θ and consequently $\mathcal{F}Z_\theta^2$ is the smallest invariant subspace for S containing $\mathcal{F}\rho_\theta$. It follows that m_θ coincides with the inner factor of $\mathcal{F}\rho_\theta$. Formula (1.5) clearly shows that $\mathcal{F}\rho_\theta$ is analytic across the arc $\{\mu : |\mu| = 1, \mu \neq 1\}$. Indeed, this follows from the fact that $\zeta(s)$ is analytic across the line $\operatorname{Re} s = \frac{1}{2}$. Then a simple property of inner functions (cf. theorem II. 6.3 from [5]) implies that m_θ is also analytic across $\{\mu : |\mu| = 1, \mu \neq 1\}$ and hence it has the form

$$m_\theta(\mu) = B_\theta(\mu)\exp\left(\tau\frac{\mu + 1}{\mu - 1}\right)$$

where B_θ is a Blaschke product and $\tau \geqq 0$. If $\tau \neq 0$, it would follow that $\lim_{\mu \uparrow 1}(\mathcal{F}\rho_\theta)(\mu) = 0$, or we have

$$\lim_{\mu \uparrow 1}(\mathcal{F}\rho_\theta)(\mu) = \lim_{s \to \infty}\frac{\alpha}{\sqrt{2}}\frac{s + \frac{1}{2}}{s}(\theta - \theta^s)\zeta(s) = \frac{\alpha\theta}{\sqrt{2}} \neq 0.$$

We conclude that $\tau = 0$ and m_θ is the Blaschke product formed with the zeros of $\mathcal{F}\rho_\theta$. The zeros of $\mathcal{F}\rho_\theta$ are $\{\mu : s = (1+\mu)/(1-\mu)+\frac{1}{2} \in A_\theta\}$ and the multiplicity of μ coincides with $n(s)$, $s = (1+\mu)/(1-\mu)+\frac{1}{2}$. By the canonical factorization theorem for H^2 functions we have

$$\mathcal{F}Z_\theta^2 = \left\{ g \in H^2 : g(\mu) = O((\mu-\mu_0)^{n(s_0)}), \quad s_0 = \frac{1+\mu_0}{1-\mu_0} + \tfrac{1}{2} \in A_\theta \right\}$$

or, equivalently

$$Z_\theta^2 = \left\{ f \in L^2(0,1) : (\mathcal{F}f)(\mu) = O((\mu-\mu_0)^{n(s_0)}), \quad s_0 = \frac{1+\mu_0}{1-\mu_0} + \tfrac{1}{2} \in A_\theta \right\}.$$

By (1.4) the relations $(\mathcal{F}f)(\mu) = O((\mu-\mu_0)^n)$ and $\int_0^1 f(x)x^{s-1}dx = O((s-s_0)^n)$ are equivalent for $s = (1+\mu)/(1-\mu)+\frac{1}{2}$. The proposition follows.

Since C^2 is the space generated by $\{Z_\theta^2 : 0 < \theta < 1\}$ we easily infer the following result.

Corollary 2.2. $C^2 = \{f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{n(s_0)})$ *for all* s_0 *with* $\zeta(s_0) = 0$, $\operatorname{Re} s_0 > \frac{1}{2}\}$.

Corollary 2.3. *Let* $M \subset L^2(0,1)$ *be a subspace, invariant under* $V_2(t)$ *for* $t \geqq 0$, *such that* $M \supset Z_\theta^2$. *Then there exists a nonnegative integer valued function* ν *on* A_θ *such that* $\nu(s) \leqq n(s)$, $s \in A_\theta$, *and*

$$M = \left\{ f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{\nu(s_0)}) \quad \text{for } s_0 \in A_\theta \right\}.$$

Moreover, for every s *in* A_θ *with* $\nu(s) \neq 0$, *there exists a nonzero function* g_s *in* $L^2 \ominus M$ *such that*

$$V_2(t)g_s - \exp(t(\tfrac{1}{2}-s))g_s \in M, \qquad t \geqq 0.$$

Proof. As in the preceding proof, $\mathcal{F}M = mH^2$ for some inner function m. Since $mH^2 \supset m_\theta H^2 = \mathcal{F}Z_\theta^2$, m must be a divisor of m_θ and hence m is a Blaschke product with zeros included in the set $\{\mu : s = (1+\mu)/(1-\mu)+\frac{1}{2} \in A_\theta\}$. Moreover, if $\nu(s)$ denotes the multiplicity in m of the zero $\mu = (2s-3)/(2s+1)$, we necessarily have $\nu(s) \leqq n(s)$. The description of the subspace M follows from (1.4), as in the preceding proof.

To prove the last part of the statement, choose $s = (1+\mu)/(1-\mu)+\frac{1}{2} \in A_\theta$ with $\nu(s) \neq 0$ and note that, by a theorem of Moeller [6] (cf. also [8], Ch. III), μ is an eigenvalue of the operator $S(m)$ defined on $\mathcal{H}(m) = H^2 \ominus mH^2$ by $S(m)f = P_{\mathcal{H}(m)}Sf$, $f \in \mathcal{H}(m)$. Thus there exists $f_s \in \mathcal{H}(m)$, $f_s \neq 0$, for which

$(S(m)-\mu)f_s = 0$ or, equivalently, $(S-\mu)f_s \in mH^2$. This implies that

$$\left[S(t)-\exp\left(t\frac{\mu+1}{\mu-1}\right)\right]f_s \in mH^2.$$

Indeed, there exists a function h in $H^\infty(\|h\|_\infty \leqq 2/(1-|\mu|))$ such that

$$\exp\left(t\frac{\lambda+1}{\lambda-1}\right)-\exp\left(t\frac{\mu+1}{\mu-1}\right) = g(\lambda)(\lambda-\mu), \qquad |\lambda|<1,$$

so that

$$\left[S(t)-\exp\left(t\frac{\mu+1}{\mu-1}\right)\right]f_s = g\cdot(S-\mu)f_s \in mH^2.$$

It suffices now to set $\mathcal{F}^* f_s \in L^2 \ominus \mathcal{F}^*(mH^2) = L^2 \ominus M$. The corollary is proved.

Observe that Corollary 2.2 implies a corollary similar to Corollary 2.3, with C^2 in the place of Z_θ^2. The following result is basic in our restatement of Riemann's hypothesis. For $1 \leqq p \leqq 2$ we set $B^p = \{s : \zeta(s) = 0, \operatorname{Re} s > 1/p\}$ and

$$A_\theta^p = B^p \cup \left\{1 + \frac{2k\pi i}{\log\theta} : k \in \mathbf{Z}\setminus\{0\}\right\};$$

obviously $A_\theta^2 = A_\theta$.

THEOREM 2.4. *For $1 \leqq p \leqq 2$ and $0 < \theta < 1$ we have*

$$Z_\theta^p \cap L^2(0,1)$$
$$= \left\{f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{n(s_0)}) \quad \textit{for all } s_0 \in A_\theta^p\right\}$$

and

$$C^p \cap L^2(0,1)$$
$$= \left\{f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{n(s_0)}) \quad \textit{for all } s_0 \in B^p\right\}.$$

PROOF. The proofs of the two identities are similar so that we only prove the first one. It is clear that $Z_\theta^p \supset Z_\theta^2$ so that $Z_\theta^p \cap L^2(0,1) \supset Z_\theta^2$. Moreover, the space $Z_\theta^p \cap L^2(0,1)$ is invariant under $V(t)$, $t \geqq 0$, so that Corollary 2.3 implies the existence of a function $\nu(s) \leqq n(s)$ such that

$$Z_\theta^p \cap L^2(0,1) = \left\{f \in L^2(0,1) : \int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{\nu(s_0)}),\ s_0 \in A_\theta\right\}.$$

We note now that the functions $t^{s-1}(\log t)^k$, $k = 0, 1, 2, \ldots$, belong to $L^q(0,1)$ (where $q = p/(p-1)$ is the conjugate exponent of p) if and only if $\operatorname{Re} s > 1/p$ ($\operatorname{Re} s \geqq 1$ and $k = 0$ for $p = 1$). Since the relation

$$\int_0^1 f(x)x^{s-1}dx = O((s-s_0)^{n(s_0)}) \tag{2.1}$$

is satisfied for all f in Z_θ^2 and s_0 in A_θ, and since Z_θ^p coincides with the closure of Z_θ^2 in $L^p(0,1)$, it follows that (2.1) holds for all f in Z_θ^p and s_0 in A_θ^p. Therefore, we have $\nu(s_0) = n(s_0)$ for every s_0 in A_θ^p. In order to finish the proof we must prove that $\nu(s) = 0$ for s in $A_\theta \setminus A_\theta^p$. Assume that $\nu(s) \neq 0$ for some s in $A_\theta \setminus A_\theta^p$. By Corollary 2.3, we can find a function g_s,

$$g_s \in L^2(0,1) \ominus [Z^p \cap L^2(0,1)]$$

satisfying the relations

$$V(t)g_s - \exp[t(\tfrac{1}{2} - s)]g_s \in Z_\theta^p \tag{2.3}$$

for $t \geqq 0$. This relation implies that

$$\inf\{\| V(t)g_s - f\|_p : f \in Z_\theta^p\} = \exp[t(\tfrac{1}{2} - \operatorname{Re} s)] \inf\{\| g_s - f\|_p : f \in Z_\theta^p\} \tag{2.4}$$

On the other side, we have

$$\inf\{\| V(t)g_s - f\|_p : f \in Z_\theta^p\} \leqq \inf\{\| V(t)(g_s - h)\|_p : h \in Z_\theta^p\}$$
$$= \exp[t(\tfrac{1}{2} - 1/p)] \inf\{\| g_s - h\|_p : h \in Z_\theta^p\}$$

and a comparison of this relation with (2.4) yields $\exp(\frac{1}{2} - \operatorname{Re} s) \leqq \exp(\frac{1}{2} - 1/p)$ and therefore $\operatorname{Re} s \geqq 1/p$. Using (2.2) we get $\operatorname{Re} s = 1/p$. Note that the proof is finished in case $p = 1$. Indeed, by the theorem of Hadamard and de la Vallée–Poussin quoted above, there are no zeros s of ζ on the line $\operatorname{Re} s = 1$ and the equality $\nu(s) = 0$, $s \in A_\theta \setminus A_\theta^1$, follows for $p = 1$. We may therefore restrict ourselves to the case $p > 1$. In this case $L^p(0,1)$ is a uniformly convex space (cf. [4]) so that we can find a unique h_s in $g_s + Z_\theta^p$ satisfying the relation

$$\| h_s\|_p = \inf\{\| g_s - f\|_p : f \in Z_\theta^p\}.$$

We observe now that

$$\exp[t(s - \tfrac{1}{2})] V(t)h_s = g_s + \exp[t(s - \tfrac{1}{2})][V(t)g_s - \exp[t(\tfrac{1}{2} - s)]g_s]$$
$$+ \exp[t(s - \tfrac{1}{2})] V(t)(h_s - g_s)$$

so that relation (2.3) and the invariance of Z_θ^p under $V(t)$ imply that

$$\exp[t(s-\tfrac{1}{2})]V(t)h_s \in g_s + Z_\theta^p.$$

Moreover, by Lemma 1.1,

$$\begin{aligned}\|\exp[t(s-\tfrac{1}{2})]V(t)h_s\|_p &= \exp[t(\operatorname{Re} s-\tfrac{1}{2})]\|V(t)h_s\|_p\\ &= \exp[t(\operatorname{Re} s-\tfrac{1}{2})]\exp[t(\tfrac{1}{2}-1/p)]\|h_s\|_p\\ &= \|h_s\|_p\end{aligned}$$

and the uniqueness of h_s in $g_s + Z_\theta^p$ implies

$$\exp[t(s-\tfrac{1}{2})]V(t)h_s = h_s \qquad \text{in } L^p(0,1).$$

But this last identity is only possible when $h_s = 0$ in $L^p(0,1)$, i.e., when $g_s \in Z_\theta^p$ in contradiction with the choice of g_s in $L^2(0,1)\ominus[Z_\theta^p \cap L^2(0,1)]$. This contradiction shows that $\nu(s)=0$ for s in $A_\theta \setminus A_\theta^p$ and thus finishes the proof.

We are now ready for the main result of this paper.

THEOREM 2.5. *The following assertions are equivalent for* $1 \leqq p \leqq 2$:
(i) *There are no zeros s of ζ with* $\operatorname{Re} s > 1/p$;
(ii) $C^p = L^p(0,1)$;
(iii) $Z_\theta^p \cap L^2(0,1) = Z_\theta^1 \cap L^2(0,1)$ *for some θ with* $0<\theta<1$;
(iii)′ $Z_\theta^p \cap L^2(0,1) = Z_\theta^1 \cap L^2(0,1)$ *for all θ with* $0<\theta<1$;
(iv) $Z_\theta^p = Z_\theta^1 \cap L^p(0,1)$ *for some θ with* $0<\theta<1$;
(iv)′ $Z_\theta^p = Z_\theta^1 \cap L^p(0,1)$ *for all θ with* $0<\theta<1$.

PROOF. The equivalence of (i), (iii) and (iii)′ obviously follows from Theorem 2.4. Theorem 2.4 also implies the equivalence of (i) with the identity

$$C^p \cap L^2(0,1) = C^1 \cap L^2(0,1) = L^2(0,1)$$

which is clearly equivalent to (ii) because C^p is closed in $L^p(0,1)$. For the equivalence between (iii) and (iv) we first note that if (iv) holds then obviously

$$Z_\theta^p \cap L^2(0,1) = [Z_\theta^1 \cap L^p(0,1)] \cap L^2(0,1) = Z_\theta^1 \cap L^2(0,1)$$

so that (iii) holds. Conversely, if (iii) holds we can use the fact that Z_θ^p is the closure of $Z_\theta^p \cap L^2(0,1)$ in $L^p(0,1)$ to show that Z_θ^p contains the closure of $Z_\theta^1 \cap L^2(0,1)$ in $L^p(0,1)$, and this clearly implies (iv). The equivalence of (iii)′ and (iv)′ is proved analogously. The theorem follows.

Let us note the equivalence of (i) and (ii) constitutes the relevant part of Beurling's theorem [2].

3. Another description of Z^p_θ

For a function f in $L^p(0,1)$, $1 \leqq p < \infty$, and a number s in $[0,1]$ we denote by f_s the function defined by

$$f_s(x) = f(x/s), \qquad x \in [0,s],$$
$$= 0, \qquad x \in (s,1].$$

It is clear that the closed linear subspace of $L^p(0,1)$ generated by $\{f_s : 0 \leqq s \leqq 1\}$ coincides with the closed linear subspace generated by $\{V(t)f : t \geqq 0\}$. It is also easy to see that f_s depends continuously (in L^p) on s. This is a consequence of the density in $L^p(0,1)$ of continuous functions with compact support in $[0,1)$.

LEMMA 3.1. *For f in $L^p(0,1)$, $1 \leqq p < \infty$, the closed subspace of $L^p(0,1)$ generated by $\{f_s : 0 \leqq s \leqq 1\}$ coincides with the closure in $L^p(0,1)$ of the set of all functions g that can be written as $g = \int_0^1 f_s dF(s)$ for some continuous function F of bounded variation on $(0,\infty)$ such that $F(x) = 0$ for $x \geqq 1$.*

PROOF. Let us denote by BV the Banach space of all right-continuous functions F of bounded variation on $(0, +\infty)$ such that $F(x) = 0$ for $x \geqq 1$, with the norm $\|F\|_{BV}$ given by the total variation. It is clear that

$$\left\| \int_0^1 f_s dF(s) \right\|_p \leqq \sup\{\|f_s\|_p \cdot \|F\|_{BV} : 0 \leqq s \leqq 1\} = \|f\|_p \cdot \|F\|_{BV}.$$

Since any function in BV is the limit, in the BV norm, of a sequence of jump functions, we infer $\int_0^1 f_s dF(s)$ belongs to the closed subspace of $L^p(0,1)$ generated by $\{f_s : 0 \leqq s \leqq 1\}$. On the other hand, since the mapping $s \to f_s$ is continuous, we have

$$\lim_{\varepsilon \to 0} \left\| f_{s_0} - \int_0^1 f_s dF_\varepsilon(s) \right\|_p = 0$$

where F_ε is the continuous function which is linear on $[s_0 - \varepsilon, s_0]$, $F_\varepsilon(x) = -1$ on $[0, s_0 - \varepsilon]$ and $F_\varepsilon(x) = 0$ on (s_0, ∞). The lemma is proved.

COROLLARY 3.2. *The space Z^p_θ coincides with the closure in $L^p(0,1)$ of all functions g of the form $g = \int_0^1 (\rho_\theta)_s dF(s)$ for some continuous function F of bounded variation on $[0,\infty)$ such that $F(x) = 0$ for $x \geqq 1$.*

PROOF. It obviously follows from Lemma 3.1 for $f = \rho_\theta$.

COROLLARY 3.3. *The space $Z^p_{1/2}$ coincides with the closure in $L^p(0,1)$ of all functions g that can be written as*

$$g(x) = \sum_{k=1}^{\infty} \{F(2kx) - F((2k-1)x)\}$$

for some continuous function F of bounded variation on $[0,\infty)$ such that $F(x)=0$ for $x \geqq 1$.

PROOF. It is easy to see that

$$\rho_{1/2} = -\tfrac{1}{2}\sum_{k=1}^{\infty} \chi_{(1/2k,\,1/(2k-1)]}.$$

The corollary follows since for continuous F the function $g=\int_0^1(\rho_{1/2})_s dF(s)$ can be computed pointwise by the formula

$$\begin{aligned} g(x) &= \int_0^1 (\rho_{1/2})_s(x)dF(s) \\ &= -\tfrac{1}{2}\sum_{k=1}^{\infty} \chi_{(1/2k,\,1/(2k-1)]}(x/s)dF(s) \\ &= -\tfrac{1}{2}\sum_{k=1}^{\infty} \int_{(2k-1)x}^{2kx} dF(s) \\ &= -\tfrac{1}{2}\sum_{k=1}^{\infty} \{F(2kx)-F((2k-1)x)\}. \end{aligned}$$

REMARK 3.4. Theorem B from the introduction follows from Corollary 3.3 combined with Theorem 2.5(iv) for $\theta=\frac{1}{2}$. Indeed, the Zorn spaces Z^p from the introduction coincide with $Z^p_{1/2}$.

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